

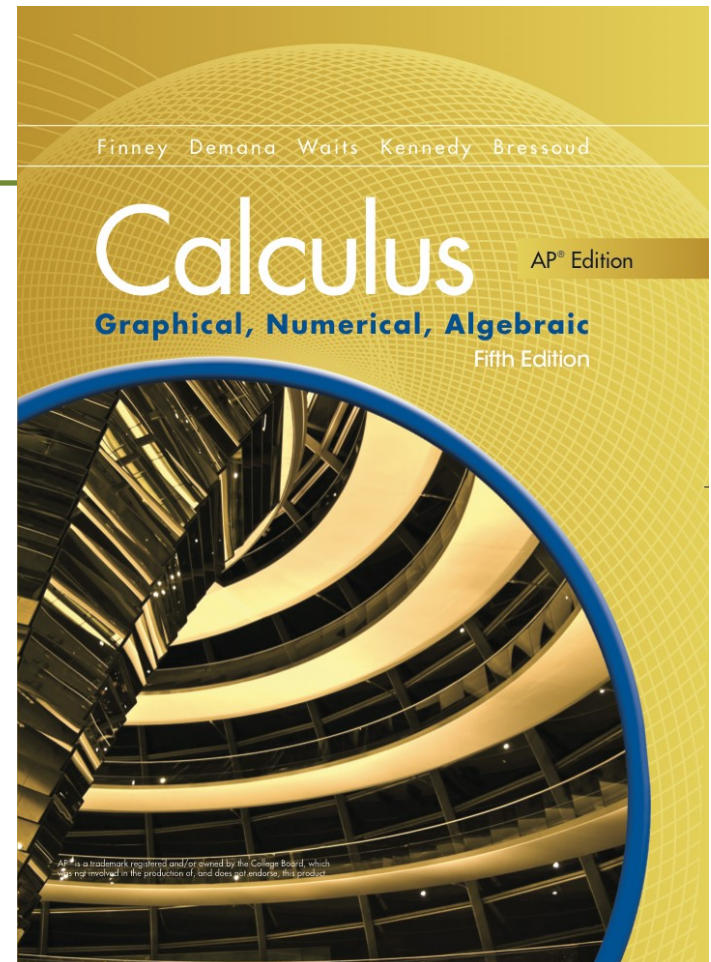
Chapter 5

Applications of Derivatives



Section 5.5

Linearization and Newton's Method



Quick Review

Find dy/dx

1. $\sin x^2$

2. $\frac{2x + \cos x}{x + 1}$

Solve the equation graphically.

3. $xe^x + 1 = 0$

4. $x^3 + 3x + 1 = 0$

5. Let $f(x) = xe^x + 1$. Write an equation for the line tangent to f at $x = 0$.

Quick Review Solutions

Find dy/dx

1. $\sin x^2$ $2x \cos x^2$

2. $\frac{2x + \cos x}{x+1}$ $\frac{2 - x \sin x - \sin x - \cos x}{(x+1)^2}$

Solve the equation graphically.

3. $xe^{-x} + 1 = 0$ $x \approx -0.567$

4. $x^3 + 3x + 1 = 0$ $x \approx -0.322$

5. Let $f(x) = xe^{-x} + 1$. Write an equation for the line tangent to f at $x=0$.

$y = x + 1$

What you'll learn about

- Linear approximation
- Sensitivity analysis
- Differentials
- Newton's method

...and why

Engineering and science depend on approximation in most practical applications; it is important to understand how approximation techniques work.

Linearization

If f is differentiable at $x = a$, then the equation of the tangent line,

$$L(x) = f(a) + f'(a)(x - a),$$

defines the **linearization of f at a** The

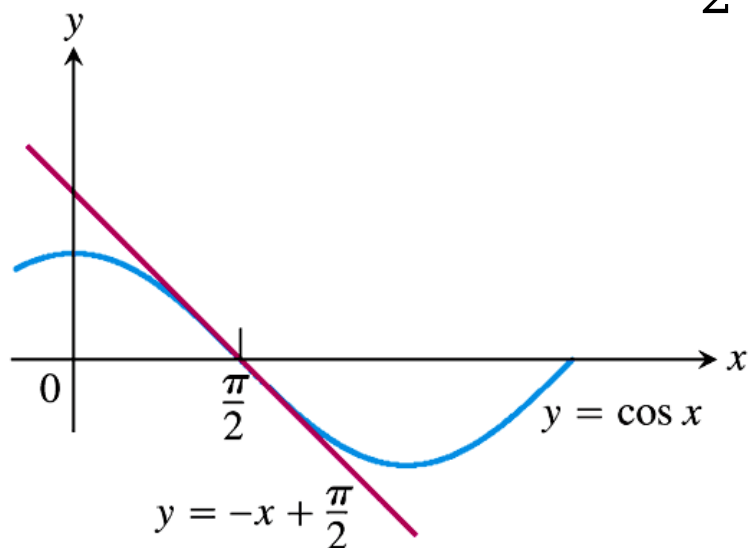
approximation $f(x) \approx L(x)$ is the **standard linear approximation of f at a** The point $x = a$ is the **center** of the approximation.

Example Finding a Linearization

Find the linearization of $f(x) = \cos x$ at $x = \pi / 2$ and use it to approximate $\cos 1.75$ without a calculator.

Since $f(\pi / 2) = \cos(\pi / 2) = 0$, the point of tangency is $(\pi / 2, 0)$. The slope of the tangent line is $f'(\pi / 2) = -\sin(\pi / 2) = -1$. Thus $L(x) = 0 + (-1)\left(x - \frac{\pi}{2}\right) = -x + \frac{\pi}{2}$.

To approximate $\cos 1.75 = f(1.75) \approx L(1.75) = -1.75 + \frac{\pi}{2}$.



Differentials

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x) dx.$$

Example Finding the Differential dy

Find the differential dy and evaluate dy for the given value of x and dx .

$$y = x^5 + 2x, \quad x = 1, \quad dx = 0.01$$

$$dy = (5x^4 + 2) dx$$

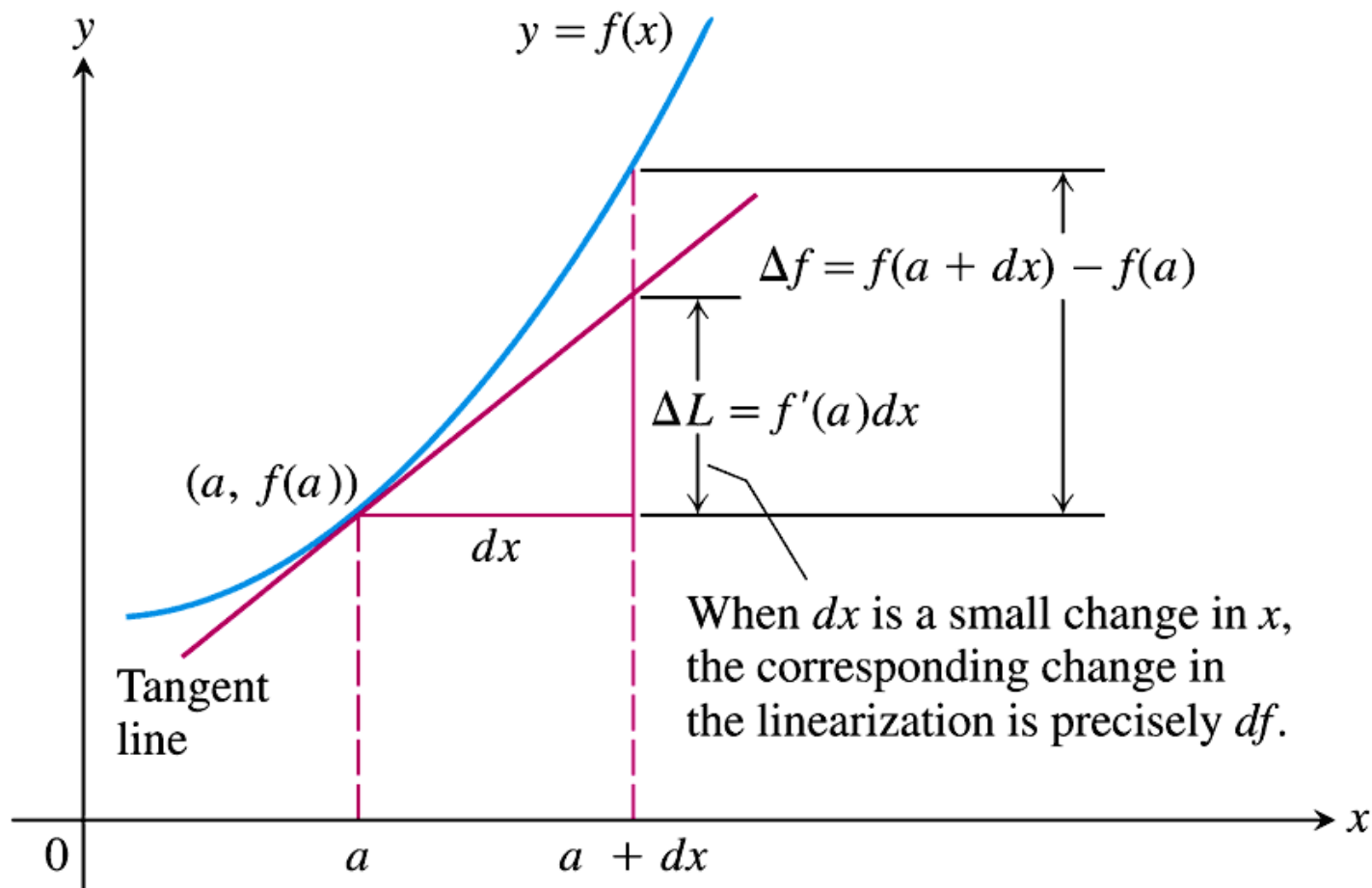
$$\begin{aligned} dy &= (5 + 2)(0.01) \\ &= 0.07 \end{aligned}$$

Differential Estimate of Change

If $f(x)$ is differentiable at $x = a$. The approximate change in the value of f when x changes from a to $a + dx$ is

$$df = f'(a) dx.$$

Approximating the Change with Differentials



Example Estimating Change with Differentials

The radius of a circle increases from $a = 5$ m to 5.1 m. Use dA to estimate the increase in the circle's area A .

Since $A = \pi r^2$, the estimated increase is

$$dA = 2\pi r dr$$

$$= 2\pi (5) (0.1)$$

$$= \pi \text{ m}^2$$

Procedure for Newton's Method

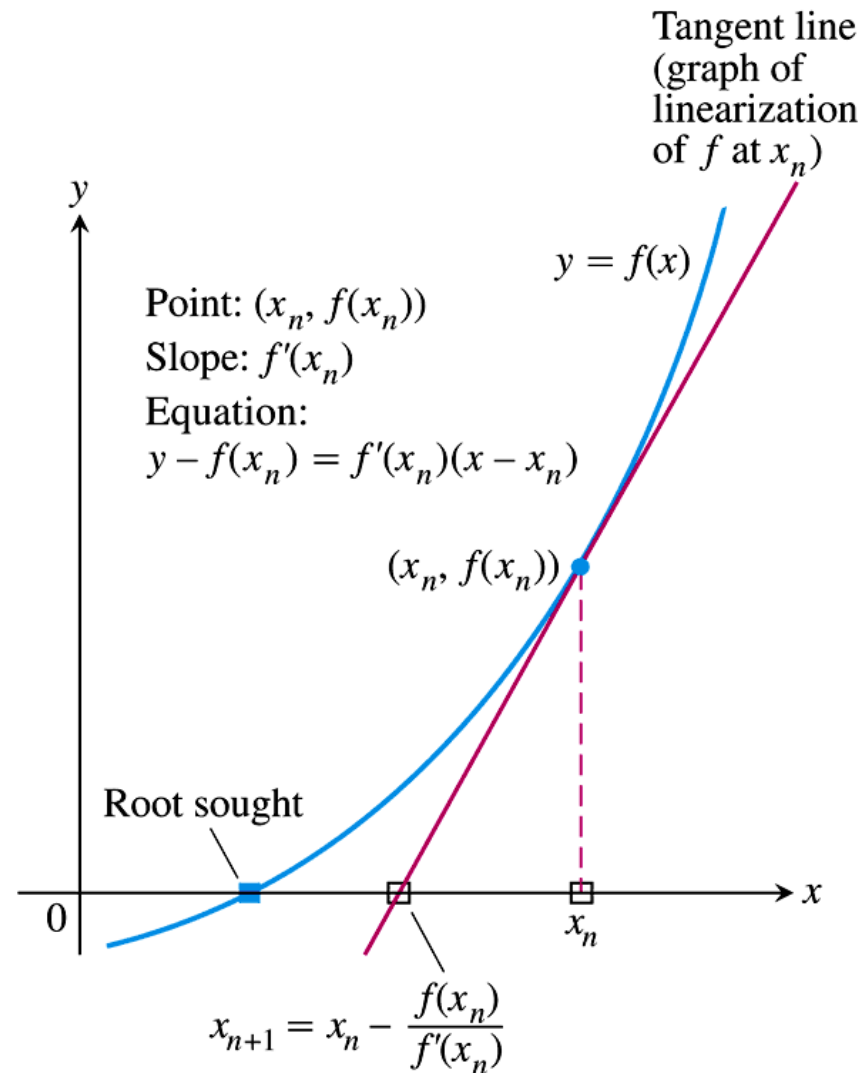
1. Guess a first approximation to a solution of the equation $f(x) = 0$.

A graph of $y = f(x)$ may help.

2. Use the first approximation to get a second, the second to get a third,

and so on, using the formula $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Procedure for Newton's Method



Using Newton's Method

Use Newton's method to solve $x^3 + 3x + 1 = 0$.

Let $f(x) = x^3 + 3x + 1$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

A graph suggests that $x_1 = -0.3$ is a good first approximation. Then,

$$x_1 = -0.3$$

$$x_2 = -0.322324159$$

$$x_3 = -0.3221853603$$

$$x_4 = -0.3221853546 \quad \text{The } x_n \text{ for } n \geq 5 \text{ all appear to equal } x_4 \text{ on the calculator.}$$

The solution to $x^3 + 3x + 1 = 0$ is about -0.3221853546 .